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Feynman Path Integral and Ordering Rules on Discrete Finite Space

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The Feynman path integral is constructed for systems whose configuration space is a discrete finite set. The construction is based on the operator formulation of quantum mechanics on a finite discrete space. We derive connections between operators and introduce the analogue of the *-multiplication for discrete symbols.

1. INTRODUCTION

The two most frequently used formulations of quantum mechanics are the operator formalism on a Hilbert space and the Feynman path integral (Feynman and Hibbs, 1955). The ambiguity of ordering operator products corresponds to the ambiguity of choosing points from each interval in which the action is evaluated (Dowker, 1976; Bertrand and Irac, 1979; Berezin, 1980). The formulation of finite-dimensional quantum mechanics has been made in several papers (Gudder and Naroditski, 1980; Šťovíček and Tolar, 1984; Santhanam, 1977; Balian and Itzykson, 1986). Moreover, in Pearle (1973) and Šťovíček (1980) the Feynman path integral was established, but only for the Rivier ordering rule. In the present paper we present the Feynman path integral and the corresponding Weyl formulation of a discrete quantum mechanics for other possible symmetrizations.

2. QUANTUM MECHANICS ON DISCRETE FINITE SPACE

For the sake of simplicity we shall restrict our attention to one classical degree of freedom. Theories for more degrees of freedom can be obtained as a tensor product of theories of one degree.

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Let $q(i)$ take one of M discrete values $\{q_i\}$, $i=0, 1, \ldots, M-1$. With each value of q_i we can connect a vector $|i\rangle$ of an orthonormal basis of M-dimensional Hilbert space $\mathscr{H}$. Let $\hat{T}$ be the corresponding map, $\hat{T}: q_i \mapsto |i\rangle$. Then we define a *position operator* (Gudder and Naroditski, 1980; Šťovíček and Tolar, 1984)

$$\hat{Q}=\sum_{j=0}^{M-1} j|j\rangle\langle j|$$

The eigenvectors of $\hat{Q}$ form a basis of the Hilbert space $\mathscr{H}$, $\{|i\rangle\}$, and i are the corresponding eigenvalues. We have $(|j\rangle)_i=\delta_{i,j}$ in this basis.

In order to obtain *momentum operators*, we close the set $\{i\}$ into the periodic chain, i.e., we get the conditions

$$|j\rangle=|j+M\rangle$$

and introduce unitary transition operators. The one-step transition operator will transform the vectors $|j\rangle$, $\hat{U}(1):|j\rangle \mapsto |j+1\rangle$, modulo M, or in matrix form,

$$\hat{U}(1)=\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ . & . & . & . & \cdots & . \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

The powers of $\hat{U}(1)$ generate cyclic matrices $\hat{U}(k)=(\hat{U}(1))^k$, $\hat{U}(k)|j\rangle=|j+k\rangle$, and $(\hat{U}(1))^M=\hat{I}d$. They provide the regular representation of the cyclic group C_M in the $\mathscr{H}$.

Let us now define a *momentum operator* $\hat{P}$ with eigenvalues p_k and eigenvectors $|p_k\rangle$ in a similar way as in continuous case (Gudder and Naroditski, 1980), i.e., as a generator of a one-parameter group of unitary transformations

$$\hat{V}(d)=e^{-id\hat{P}}$$

Since $\hat{Q}$ has a discrete spectrum of eigenvalues, d has to satisfy the conditions

$$\hat{V}(d)|j\rangle=|j+m\rangle \qquad \text{for all} \quad j(\text{mod } M) \tag{1}$$

and will depend on $m=0, 1, \ldots, M-1$. Using the resolutions of the identity

$$\hat{I}d=\sum_{k=0}^{M-1} |p_k\rangle\langle p_k|, \qquad \hat{I}d=\sum_{r=0}^{M-1} |r\rangle\langle r|$$

we get

$$\hat{V}(d)|j\rangle = \sum_{p_k} e^{-idp_k}|p_k\rangle\langle p_k|j\rangle = \sum_{p_k}\sum_{r} e^{-idp_k}|r\rangle\langle r|p_k\rangle\langle p_k|j\rangle$$

and from the conditions (1) it follows that

$$\sum_{p_k} e^{-idp_k}\langle r|p_k\rangle\langle p_k|j\rangle = \delta_{r,j+m}$$

But $\delta_{r,j+m}$ can be expressed as

$$\delta_{r,j+m} = \frac{1}{M}\sum_{k=0}^{M-1} e^{(2\pi i/M)k(r-j-m)}$$

Comparing the last two equations, after separation of r- and p_k-dependent factors, one can identify

$$p_k = k, \qquad d = \frac{2\pi}{M}m, \qquad \langle r|p_k\rangle = \frac{1}{\sqrt{M}} e^{(2\pi i/M)kr} \tag{2}$$

Hence

$$\hat{V}\left(\frac{2\pi}{M}\right) = \hat{U}(1)$$

Denoting eigenvectors of $\hat{P}$ by $|k\rangle$, where $k = 0, \ldots, M-1$, we get from (2)

$$|k\rangle = \frac{1}{\sqrt{M}}\sum_{j} e^{(2\pi i/M)kj}|j\rangle \tag{3}$$

Equation (3) describes a discrete Fourier transformation of the eigenvectors $|j\rangle$. Matrix elements of the operator $\hat{P}$ in the basis of eigenvectors of $\hat{Q}$ are

$$\langle m|\hat{P}|n\rangle = \frac{1}{M}\sum_{k} k\, e^{(2\pi i/M)k(m-n)} = \begin{cases} \frac{1}{2}(M-1) & \text{if } m = n \\ (e^{(2\pi i/M)(m-n)} - 1)^{-1} + M^{-1} & \text{otherwise} \end{cases}$$

and the matrix elements of the commutator are

$$\langle m|[\hat{Q}, \hat{P}]|n\rangle = (m-n)\langle m|\hat{P}|n\rangle$$

Although the commutation relations are different from the continuous case, the Weyl relations do hold in the discrete case (Gudder and Naroditski, 1980):

$$\begin{aligned} e^{(2\pi i/M)t\hat{Q}}\, e^{(2\pi i/M)s\hat{P}}|j\rangle &= e^{(2\pi i/M)t\hat{Q}}|j-s) \\ &= e^{(2\pi i/M)t(j-s)}|j-s\rangle \\ &= e^{-(2\pi i/M)ts}\, e^{(2\pi i/M)s\hat{P}}\, e^{(2\pi i/M)tj}|j\rangle \\ &= e^{-(2\pi i/M)ts}\, e^{(2\pi i/M)s\hat{P}}\, e^{(2\pi i/M)t\hat{Q}}|j\rangle \end{aligned}$$

Hence the Heisenberg uncertainty principle holds in the following weaker sense:

If the system is in the eigenstate $|j\rangle$ of the operator $\hat{Q}$, then the probability of measuring each eigenvalue of $\hat{P}$ is the same.

Let the system be in the state $|j\rangle$; then the probability of finding the system in an eigenstate $|k\rangle$ of $\hat{P}$ is

$$|\langle j|k\rangle|^2 = \langle j|k\rangle\langle k|j\rangle = \frac{1}{M}$$

The reversible time evolution of any isolated quantum system is determined by a strongly continuous one-parameter group $\hat{L}(t)$ of unitary operators acting in a Hilbert space. According to Stone's theorem, there exists a self-adjoint operator $\hat{H}$, the *Hamiltonian*, such that

$$\hat{L}(t) = e^{-i\hat{H}t}$$

3. THE SYMBOLS AND THE ORDERING RULES

Let us assume now that the Hamiltonian $\hat{H}$ is expressible as a general function $\hat{H}(\hat{P}, \hat{Q})$ of the operators $\hat{P}$ and $\hat{Q}$. The problem of symmetrization in the discrete case is different from this problem in the continuous case. Due to the commutation relations, it is not possible to separate $\hat{P}$- and $\hat{Q}$-dependent factors in

$$e^{(2\pi i/M)(k\hat{P}+m\hat{Q})}$$

However, we can try to formulate a discrete analogue of the Weyl–Wigner general correspondence rule in a continuous space. With every operator $\hat{H}(\hat{P}, \hat{Q})$ we associate a real matrix $h_{p,q}$—its *symbol*—on the discrete phase space $C_M \times C_M$:

$$\hat{H}(\hat{P}, \hat{Q}) = \frac{1}{M^2} \sum_k \sum_m \sum_p \sum_q e^{(i\pi/M)km} \times e^{(2\pi i/M)m\hat{Q}} e^{(2\pi i/M)k\hat{P}} e^{-(2\pi i/M)(kp+mq)} f_{k,m} h_{p,q} \tag{4}$$

The complex matrix $f_{k,m}$ is restricted by two conditions:

(a) In order for $\hat{Q}$ to have the symbol q and $\hat{P}$ the symbol p, $f_{k,0} = f_{0,m} = 1$ must hold.
(b) Since the operator $\hat{H}$ has to be Hermitian, we demand $f_{k,m} = f^{+}_{m-k,m-k}$.

We can express matrix elements of $\hat{H}(\hat{P}, \hat{Q})$ in the basis $\{|j\rangle\}$,

$$\begin{aligned}\langle r|\hat{H}(\hat{P}, \hat{Q})|s\rangle &= \sum_l \langle r|\hat{H}|l\rangle\langle l|s\rangle \\ &= \times \frac{1}{M^3} \sum_l \sum_k \sum_m \sum_p \sum_q e^{(i\pi/M)km} \\ &\quad \times e^{(2\pi i/M)[m(r-q)+k(l-p)+l(r-s)]} f_{k,m} h_{p,q} \qquad (5)\end{aligned}$$

and perform the summations over l and k to get

$$\langle r|\hat{H}(\hat{P}, \hat{Q})|s\rangle = \frac{1}{M^2} \sum_p \sum_q C_{r,s;p,q} h_{p,q} \qquad (6)$$

where

$$C_{r,s;p,q} = \sum_m e^{(2\pi i/M)[p(r-s)+(m/2)(r+s-2q)]} f_{s-r,m}$$

In analogy with the continuous case, we can introduce symmetrizations according to Table I.

Let us now separate the real and the imaginary parts of matrix elements $\langle r|\hat{H}(\hat{P}, \hat{Q})|s\rangle$; $H^R_{r,s}$ denotes the real part if $r \geq s$ and the imaginary part if $r < s$. Equation (6) is transformed into M^2 linear equations with M^2 real variables

$$H^R_{r,s} = \frac{1}{M^2} \sum_{p,q} C^R_{r,s;p,q} h_{p,q}$$

If the rank of $C^R_{r,s;p,q}$ is $M^2 - n$, then this equation has solutions only for the operators which are restricted to satisfy n conditions, and we get n free parameters in $h_{p,q}$. When an inverse matrix of $C^R_{r,s;p,q}$ exists (i.e., its rank is M^2), we can associate the multiplication of operators

$$\hat{H} = \hat{D}\hat{G}$$

Table I

Symmetrization	$f_{r,s}$
Weyl–McCoy	1
Born–Jordan	$\{\sin[(\pi/M)rs]\}/(\pi/M)rs$
Rivier	$\cos(\pi/M)rs$

with the $*$-product of symbols

$$h_{p,q} = \frac{1}{M^2} C^{-1}_{r,s;p,q} C_{r,t;k,l} C_{t,s;m,n} d_{k,l} g_{m,n} = (d*g)_{p,q}$$

which is the analogue of the $*$-product in the continuous case. Some transformation matrices and values of determinants for low values of M and for different orderings are shown in the Appendix.

4. THE FEYNMAN PATH INTEGRAL IN A DISCRETE SPACE

Let us derive the Feynman path integral in a discrete space, following the continuous case as closely as possible. Let $|q't'\rangle$ and $|q_0 t_0\rangle$ be the state vectors of the final state and of the initial state at times t' and t_0. We are looking for the transition amplitude between these two states. With the time interval $t' - t_0$ divided into N intervals of duration $\varepsilon = (t' - t_0)/N$, the transition amplitude is

$$\begin{aligned} \langle q't' | q_0 t_0 \rangle &= \langle q' | \, e^{-(i/\hbar)\hat{H}(t'-t_0)} | q_0 \rangle = \sum_{q_1} \cdots \sum_{q_{N-1}} \\ &\times \langle q' | \, e^{-(i/\hbar)\hat{H}\varepsilon} | q_{N-1} \rangle \langle q_{N-1} | \, e^{-(i/\hbar)\hat{H}\varepsilon} | q_{N-2} \rangle \\ &\times \langle q_{N-2} | \cdots | q_1 \rangle \langle q_1 | \, e^{-(i/\hbar)\hat{H}\varepsilon} | q_0 \rangle \end{aligned}$$

where $q_k = q(t_0 + k\varepsilon)$. Each factor on the right-hand side can be expressed to first order in ε

$$\langle q_{k+1} | \, e^{-(i/\hbar)\hat{H}\varepsilon} | q_k \rangle = \langle q_{k+1} | q_k \rangle - \frac{i}{\hbar} \varepsilon \langle q_{k+1} | \hat{H} | q_k \rangle$$

By using the resolution of the identity

$$\hat{Id} = \sum_{p_k=0}^{M-1} |p_k\rangle\langle p_k|$$

it is easy to show that

$$\begin{aligned} \langle q_{k+1} | \, e^{-(i/\hbar)\hat{H}\varepsilon} | q_k \rangle &= \sum_{p_k=0}^{M-1} \langle q_{k+1} | p_k \rangle \langle p_k | q_k \rangle \\ &\quad - \frac{i}{\hbar} \varepsilon \frac{1}{M^2} \sum_p \sum_q \sum_j e^{(2\pi i/M)p(q_{k+1} - q_k)} \\ &\quad \times e^{(2\pi i/M)j((q_{k+1} + q_k)/2 - q)} f_{q_k - q_{k+1}, j} h_{p,q} \end{aligned}$$

Writing p_k for p in the second sum and using (2), we get to first order in ε

$$\langle q_{k+1}|\, e^{-(i/\hbar)\hat{H}\varepsilon}|q_k\rangle = \frac{1}{M}\sum_{p_k} e^{(2\pi i/M)p_k(q_{k+1}-q_k)}$$

$$\times\left[1 - \frac{i}{\hbar}\,\varepsilon\,\frac{1}{M}\sum_q\sum_j e^{(2\pi i/M)j((q_{k+1}+q_k)/2-q)} f_{q_k-q_{k+1},j}\, h_{p_k,q}\right]$$

$$= \frac{1}{M}\sum_{p_k} e^{(2\pi i/M)p_k(q_{k+1}-q_k)}\left[1 - \frac{i}{\hbar}\,\varepsilon h'(p_k, q_{k+1}, q_k)\right]$$

where

$$h'(p_k, q_{k+1}, q_k) = \frac{1}{M}\sum_q\sum_j e^{(2\pi i/M)j(q_{k+1}+q_k-2q)/2} f_{q_k-q_{k+1},j}\, h_{p_k,q}$$

Since for small ε we can write

$$\langle q_{k+1}|\, e^{-(i/\hbar)\hat{H}\varepsilon}|q_k\rangle = \frac{1}{M}\sum_{p_k} e^{(2\pi i/M)p_k(q_{k+1}-q_k)-(i/\hbar)\varepsilon h'(p_k,q_{k+1},q_k)}$$

the final form of transition amplitude is

$$\langle q't'|q_0t_0\rangle = \lim_{N\to\infty,\,\varepsilon N=t'-t_0} \frac{1}{M^N}\sum_{q_{N-1}}\cdots\sum_{q_1}\sum_{p_{N-1}}\cdots\sum_{p_0}\sum_k$$
$$\times\, e^{(2\pi i/M)p_k(q_{k+1}-q_k)-(i/\hbar)\varepsilon h'(p_k,q_{k+1},q_k)} \tag{7}$$

In order to establish the Feynman sum on discrete phase space, we start with the Feynman prescription to calculate the transition amplitude (Feynman and Hibbs, 1955)

$$\langle q't'|q_0t_0\rangle = K(q't', q_0t_0) = \sum A\, e^{(i/\hbar)S[x(t)]}$$

where the sum is over all paths, A is the normalization constant, and $S[x(t)]$ the action. This is evaluated in the continuous space by means of time interval discretization,

$$K_N(q't', q_0t_0) = B^{N-1}\int dq_{N-1}\cdots\int dq_1\int dp_{N-1}\cdots\int dp_0$$

$$\times\exp\left(\frac{i}{\hbar}\sum_{k=0}^{N-1} p_k(q_{k+1}-q_k\right)\exp\left(-\frac{i}{\hbar}\int_{t_k}^{t_{k+1}} H(p(t), q(t), t)\, dt\right)$$

Using the mean-value theorem, we can write the time integral as

$$\int H(p(t), q(t), t)\, dt = \varepsilon H'(p_k, q_{k+1}, q_k)$$

where $\varepsilon = t_{k+1} - t_k$ and H' is H evaluated at any point q from the interval $q_{k+1} \geq q \geq q_k$ (Berezin, 1980). In discrete space we have sums over all possible values of coordinates and momenta instead of the integrals in continuous space, i.e.,

$$K_N(q't', q_0 t_0) = B^{N-1} \sum_{q_{N-1}} \cdots \sum_{q_1} \sum_{p_{N-1}} \cdots \sum_{p_0} \times \exp\left(\frac{i}{\hbar} \sum_{k=0}^{N-1} [p_k(q_{k+1} - q_k) - \varepsilon H'(p_k, q_{k+1}, q_k)]\right) \tag{8}$$

Comparing this equation for the transition amplitude with equation (7), which we obtained from the operator formulation, and after proper redefinition of p and q, we see that the matrix $h'(p_k, q_{k+1}, q_k)$ is a discrete Hamiltonian if and only if it is *real*. This is true only for the discrete Rivier symmetrization, which gives

$$h'(p_k, q_{k+1}, q_k) = \tfrac{1}{2}(h_{p_k, q_k} + h_{p_k, q_{k+1}})$$

If one uses the discrete analogue of Weyl–McCoy or Born–Jordan symmetrization, then a complex matrix $h'(p_k, q_{k+1}, q_k)$ is obtained. We give h' for both these symmetrizations in the case $M = 2$:

For Weyl–McCoy, $f_{r,s} = 1$ and

$$h'(p, 0, 0) = h_{p,0}, \qquad h'(p, 1, 1) = h_{p,1}$$

$$h'(p, 1, 0) = \tfrac{1}{2}[h_{p,0}(1+i) + h_{p,1}(1-i)] = h'(p, 0, 1)$$

For Born–Jordan, $f_{r,s} = \{\sin[(\pi/M)rs]\}/(\pi/M)rs$ and

$$h'(p, 0, 0) = h_{p,0}, \qquad h'(p, 1, 1) = h_{p,1}$$

$$h'(p, 1, 0) = h_{p,0}\left(1 + \frac{2i}{\pi}\right) + h_{p,1}\left(1 + \frac{2i}{\pi}\right) = h'(p, 0, 1)$$

5. CONCLUSION

The problem of operator symmetrization and the corresponding ambiguity of quantization can be approached in two main ways: (i) in the operator formalism of quantum mechanics, and (ii) via the Feynman path integral. In this paper we analyzed the transition amplitude from these points of view and compared the resulting expressions (7), (8).

APPENDIX

Matrices $C^R_{r,s;p,q}$ for $M=2$ and $M=3$.

For $M=2$ and $f_{r,s}=1$:

$$C^R_{r,s;p,q}=\begin{pmatrix} 2.000 & 0.000 & 2.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 \\ 1.000 & 1.000 & -1.000 & -1{\cdot}000 \\ 0.000 & 2.000 & 0.000 & 2.000 \end{pmatrix}$$

For $M=2$ and $f_{r,s}=\cos(\pi/M)rs$:

$$C^R_{r,s;p,q}=\begin{pmatrix} 2.000 & 0.000 & 2.000 & 0.000 \\ 1.000 & -1.000 & -1.000 & 1.000 \\ 1.000 & 1.000 & -1.000 & -1.000 \\ 0.000 & 2.000 & 0.000 & 2.000 \end{pmatrix}$$

For $M=2$ and $f_{r,s}=\{\sin[(\pi/M)rs]\}/(\pi/M)rs$:

$$C^R_{r,s;p,q}=\begin{pmatrix} 2.000 & 0.000 & 2.000 & 0.000 \\ 0.637 & -0.637 & -0.637 & 0.637 \\ 1.000 & 1.000 & -1.000 & -1.000 \\ 0.000 & 2.000 & 0.000 & 2.000 \end{pmatrix}$$

For $M=3$ and $f_{r,s}=1$:

$$C^R_{r,s;p,q}=\begin{pmatrix} 3.00 & 0.00 & 0.00 & 3.00 & 0.00 & 0.00 & 3.00 & 0.00 & 0.00 \\ 1.73 & -1.73 & 0.00 & -1.73 & 0.00 & -0.87 & 0.00 & 1.73 & 0.87 \\ 0.00 & 0.00 & 0.00 & 0.00 & 2.60 & 0.00 & 0.00 & -2.60 & 0.00 \\ 1.00 & 1.00 & 1.00 & -2.00 & 1.00 & -0.50 & 1.00 & -2.00 & -0.50 \\ 0.00 & 3.00 & 0.00 & 0.00 & 3.00 & 0.00 & 0.00 & 3.00 & 0.00 \\ 0.00 & 1.73 & -1.73 & -0.87 & -1.73 & 0.00 & 0.87 & 0.00 & 1.73 \\ 0.00 & 3.00 & 0.00 & 0.00 & -1.50 & 0.00 & 0.00 & -1.50 & 0.00 \\ 1.00 & 1.00 & 1.00 & -0.50 & -2.00 & 1.00 & -0.50 & 1.00 & -2.00 \\ 3.00 & 0.00 & 0.00 & 3.00 & 0.00 & 0.00 & 3.00 & 0.00 & 0.00 \end{pmatrix}$$

For $M=3$ and $f_{r,s}=\cos(\pi/M)rs$:

$$C^R_{r,s;p,q}=\begin{pmatrix} 3.00 & 0.00 & 0.00 & 3.00 & 0.00 & 0.00 & 3.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & -1.30 & -1.30 & 0.00 & 1.30 & 1.30 & 0.00 \\ 0.00 & 0.00 & 0.00 & 1.30 & 0.00 & 1.30 & -1.30 & 0.00 & -1.30 \\ 1.50 & 1.50 & 0.00 & -0.75 & -0.75 & 0.00 & -0.75 & -0.75 & 0.00 \\ 0.00 & 3.00 & 0.00 & 0.00 & 3.00 & 0.00 & 0.00 & 3.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & -1.30 & -1.30 & 0.00 & 1.30 & 1.30 \\ 1.50 & 0.00 & 1.50 & -0.75 & 0.00 & -0.75 & -0.75 & 0.00 & 0.75 \\ 0.00 & 1.50 & 1.50 & 0.0 & -0.75 & -0.75 & 0.00 & -0.75 & -0.75 \\ 3.00 & 0.00 & 0.00 & 3.00 & 0.00 & 0.00 & 3.00 & 0.00 & 0.00 \end{pmatrix}$$

For $M=3$ and $f_{r,s}=\{\sin[(\pi/M)rs]\}/(\pi/M)rs$:

$$C^R_{r,s;p,q}=\begin{pmatrix} 3.00 & 0.00 & 0.00 & 3.00 & 0.00 & 0.00 & 3.00 & 0.00 & 0.00 \\ 1.07 & -1.07 & 0.00 & -1.58 & -0.51 & -0.51 & 0.51 & 1.58 & 0.51 \\ 0.54 & 0.00 & -0.54 & 0.51 & 1.05 & 1.05 & -1.05 & -1.05 & -0.51 \\ 1.21 & 1.21 & 0.59 & -1.53 & 0.33 & -0.29 & 0.33 & -1.53 & -0.29 \\ 0.00 & 3.00 & 0.00 & 0.00 & 3.00 & 0.00 & 0.00 & 3.00 & 0.00 \\ 0.00 & 1.07 & -1.07 & -0.51 & -1.58 & -0.51 & 0.51 & 0.51 & 1.58 \\ 0.90 & 1.21 & 0.90 & -0.02 & -0.60 & -0.91 & -0.91 & -0.60 & 0.02 \\ 0.59 & 1.21 & 1.21 & -0.29 & -1.53 & 0.33 & -0.29 & 0.33 & -1.53 \\ 0.00 & 0.00 & 3.00 & 0.00 & 0.00 & 3.00 & 0.00 & 0.00 & 3.00 \end{pmatrix}$$

Determinants of $C^R_{r,s;p,q}$ matrices for $M=2$–11
For $f_{r,s}=1$:

$M=2$ DET: 32
$M=3$ DET: $-3.8353.45055105058$
$M=4$ DET: $-3.388131789017221E-021$
$M=5$ DET: $2.833344240073241E+018$
$M=6$ DET: $-1.310004767714637E-051$
$M=7$ DET: $-1.725829256450258E+042$
$M=8$ DET: $-7.146474664783708E-110$
$M=9$ DET: $3.342878352450051E+075$
$M=10$ DET: $-2.024665391665977E+024$
$M=11$ DET: $8.086095447601342E+162$

For $f_{r,s}=\cos(\pi/M)rs$:

$M=2$ DET: 0
$M=3$ DET: -21573.81593496589
$M=4$ DET: 0
$M=5$ DET: $2.424349759030538E+018$
$M=6$ DET: $7.174805450354139E-041$
$M=7$ DET: $-9.036415424332032E+044$
$M=8$ DET: 0
$M=9$ DET: $1.100488004757944E+087$
$M=10$ DET: $-1.985841392180757E+110$
$M=11$ DET: $5.704713054790777E+150$

For $f_{r,s}=\{\sin[(\pi/M)rs]\}/(\pi/M)rs$:

$M=2$	DET:	20.372832715
$M=3$	DET:	−22704.792753
$M=4$	DET:	2.81218290162E−007
$M=5$	DET:	1.61127097242E+016
$M=6$	DET:	−2.45850995703E−032
$M=7$	DET:	−2.06253782970E+036
$M=8$	DET:	1.30215140501E−025
$M=9$	DET:	2.34965380022E+009E
$M=10$	DET:	−2.55558945697E+082
$M=11$	DET:	7.15081831484E+115

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